1 Deferred Acceptance

Let $S$ be the set of students, $U$ the set of universities and assume each university selects at most one student and $|S| = |U|$. Let $\psi$ be the set of stable matchings and define $\text{best} : S \rightarrow U$ to be a mapping of a student to the best university he/she could match among all stable matchings in $\psi$. Similarly, we can define $\text{worst} : U \rightarrow S$ to be a mapping of a university to the worst student it could match.

**Theorem 1** When students propose, Deferred Acceptance (DA) outputs the best stable matching for students or

$$M^* = \{(s, \text{best}(s))|s \in S\}$$

**Proof.** Assume for contradiction DA outputs a matching $M \neq M^*$. During the execution of DA, let student $s$ be the first student that is rejected by $\text{best}(s)$ in favor of some student $s'$. Consider the stable matching $M'$ where $s$ matches with $\text{best}(s)$. In $M'$, student $s'$ matches with $u' \neq \text{best}(s)$. Clearly, $s' \succ_{\text{best}(s)} s$ since $\text{best}(s)$ rejects $s$ in favor of $s'$. In addition, since $s$ is the first student to be rejected by $\text{best}(s)$, $s'$ proposes to $\text{best}(s)$ before $u'$ implying $\text{best}(s) \succ_{s'} u'$. Finally, observe $(s', \text{best}(s))$ forms a blocking pair on $M'$, contradicting $M'$ is a stable matching. $\blacksquare$

**Theorem 2** When students propose, Deferred Acceptance (DA) outputs the worst possible stable matching for universities or

$$M^* = \{(\text{worst}(u), u)|u \in U\}$$

**Proof.** Let $M$ be the stable matching resulting of the execution of DA and assume towards contradiction that it is not the worst possible stable matching for universities. Then there is a university $u$ that matches with a student $s$ and $s \succ_{u} s'$, where $s' = \text{worst}(u)$. By definition, there is a stable matching $M'$ where $u$ matches with student $s'$. In the matching $M$, student $s$ matches with some university $u'$ and by Theorem $1$, $u \succ_{s} u'$. Finally, observe $(s, u)$ forms a blocking pair in $M'$, contradicting $M'$ is a stable matching. $\blacksquare$

**Corollary 3** DA always outputs the same stable matching.

**Proof.** Follows directly from Theorem $1$ or $2$. $\blacksquare$
2 Optimization

Minimize the following functions where $x^2$ represents the inner product $\langle x, x \rangle$.

1. $f(x) = ce^{-x^2}, \ x \in \mathbb{R}^n$
   
   Solution. $\nabla f(x) = -2cx e^{-x^2}$ so the function is minimized when $x \to \infty$. ■

2. $f(x) = x_1^2 - 2x_3^2, (x_1, x_2) \in \mathbb{R}^+$
   
   Solution. $\nabla f(x) = (2x_1, 4x_2)$ so the function is minimized when $x_1 = x_2 = 0$. ■

3. $f(x) = ce^{-x^2}$ and $x \in \mathbb{R}^2$ is a point in the unit circle centered in 0.
   
   Solution. The constraint states that $x_1^2 + x_2^2 = 1$ and note that $x^2 = ||x||^2 = x_1^2 + x_2^2$; therefore $f(x) = \frac{c}{e}$ for any feasible value of $x$. ■

3 Probability

1. Consider a classroom with $n$ students where each student is assigned a sit. The students leave the room and pick a sit uniformly at random. Compute the expected number of students that sit on their assigned sit.

   Solution. The probability a student picks his own sit is $\frac{1}{n}$. Let $X_i$ be the indicator for student $i$ picking his own sit and let $X = \sum_{i=1}^{n} X_i$ be the total number of students that sits on their own sit. By linearity of expectation,

   $$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 1$$

   ■

2. Consider the function $T : \{0, 1\}^n \rightarrow \mathbb{R}$. Given a random variable $x$, suppose you measure that $E(T(x)) = n^2$. What can you can conclude about $\sup_x T(x)$ and $\inf_x T(x)$?

   Solution. $E(T(x)) = n^2$ implies that $\exists x \in \{0, 1\}^n$ such that $T(x) \geq n^2$; therefore, $\sup_x T(x) \geq n^2$. Similarly, $\exists x \in \{0, 1\}^n$ such that $T(x) \leq n^2$; therefore, $\inf_x T(x) \leq n^2$. ■