COS 445 – Precept 1

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1 Deferred Acceptance

Let S be the set of students, U the set of universities and assume each university selects at most one student and $|S| = |U|$. Let ψ be the set of stable matchings and define $best : S \rightarrow U$ to be a mapping of a student to the best university he/she could match among all stable matchings in ψ. Similarly, we can define $worst : U \rightarrow S$ to be a mapping of a university to the worst student it could match.

Theorem 1 *When students propose, Deferred Acceptance (DA) outputs the best stable matching for students or*

$$M^* = \{(s, best(s))|s \in S\}$$

Proof. Assume for contradiction DA outputs a matching $M \neq M^*$. During the execution of DA, let student s be the first student that is rejected by $best(s)$ in favor of some student s'. Consider the stable matching M' where s matches with $best(s)$. In M', student s' matches with $u' \neq best(s)$. Clearly , $s' \succ_{best(s)} s$ since $best(s)$ rejects s in favor of s'. In addition, since s is the first student to be rejected by $best(s)$, s' proposes to $best(s)$ before u' implying $best(s) \succ_{s'} u'$. Finally, observe $(s', best(s))$ forms a blocking pair on M', contradicting M' is a stable matching. ∎

Theorem 2 *When students propose, Deferred Acceptance (DA) outputs the worst possible stable matching for universities or*

$$M^* = \{(worst(u), u)|u \in U\}$$

Proof. Let M be the stable matching resulting of the execution of DA and assume towards contradiction that it is not the worst possible stable matching for universities. Then there is a university u that matches with a student s and $s \succ_u s'$, where $s' = worst(u)$. By definition, there is a stable matching M' where u matches with student s'. In the matching M, student s matches with some university u' and by Theorem 1, $u \succ_s u'$. Finally, observe (s, u) forms a blocking pair in M', contradicting M' is a stable matching. ∎

Corollary 3 *DA always outputs the same stable matching.*

Proof. Follows directly from Theorem 1 or 2. ∎

2 Optimization

Minimize the following functions where x^2 represents the inner product $\langle x, x \rangle$.

1. $f(x) = ce^{-x^2}, x \in \mathbb{R}^n$

   Solution. $\nabla f(x) = -2cxe^{-x^2}$ so the function is minimized when $x \to \infty$. ■

2. $f(x) = x_1^2 - 2x_2^2$, $(x_1, x_2) \in \mathbb{R}^{+2}$.

   Solution. $\nabla f(x) = (2x_1, 4x_2)$ so the function is minimized when $x_1 = x_2 = 0$. ■

3. $f(x) = ce^{-x^2}$ and $x \in \mathbb{R}^2$ is a point in the unit circle centered in 0.

   Solution. The constraint states that $x_1^2 + x_2^2 = 1$ and note that $x^2 = ||x||^2 = x_1^2 + x_2^2$; therefore $f(x) = \frac{c}{e}$ for any feasible value of x. ■

3 Probability

1. Consider a classroom with n students where each student is assigned a sit. The students leave the room and pick a sit uniformly at random. Compute the expected number of students that sit on their assigned sit.

   Solution. The probability a student picks his own sit is $\frac{1}{n}$. Let X_i be the indicator for student i picking his own sit and let $X = \sum_{i=1}^{n} X_i$ be the total number of students that sits on their own sit. By linearity of expectation,

   $$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 1$$

   ■

2. Consider the function $T : \{0, 1\}^n \to \mathbb{R}$. Given a random variable x, suppose you measure that $E(T(x)) = n^2$. What can you can conclude about $\sup_x T(x)$ and $\inf_x T(x)$?

   Solution. $E(T(x)) = n^2$ implies that $\exists x \in \{0, 1\}^n$ such that $T(x) \geq n^2$; therefore, $sup_x T(x) \geq n^2$. Similarly, $\exists x \in \{0, 1\}^n$ such that $T(x) \leq n^2$; therefore, $\inf_x (X) \leq n^2$. ■